N/S

NASA

YN-64-TM

314415

P. 28

A LAGRANGIAN THEORY FOR TENSOR INTERACTION

G. KALMAN

Department of Physics
Boston College
Chestnut Hill, MA 02167

B. DATTA

Tata Institute of Fundamental Research
Bombay 400 005, INDIA

V. CANUTO

NASA, Goddard Institute for Space Studies
Goddard Space Flight Center
New York, NY 10025

(NASA-TM-101925) A LAGRANGIAN THEORY FOR TENSOR INTERACTION (NASA) 28 p

N90-71272

Unclas

00/64 0277478

ABSTRACT

A consistent Lagrangian theory for particles interacting through a massive tensor (spin-2) field is formulated. The theory is non-linear; the origin of non linearities is partly the same as in the massless Einstein theory, partly can be traced to the inherent field dependence of rest mass and momenta for tensor interaction. The theory reduces to the correct, ghostless Pauli-Fierz structure in the linear limit and to the Einstein theory in the massless case. The formalism is amenable to the study of the properties of high density matter in situations where the interaction mediated through the spin-2 f-mesons is important.

I. INTRODUCTION

In this paper we present a Lagrangian formalism for particles interacting through a massive tensor (spin-2) field. Our motivation for doing this is twofold. In the first place, theories of interaction of massive higher-spin fields pose certain basic problems of their own which, if not treated carefully, can lead to inconsistent and unphysical results. In contrast to scalar (spin 0) and vector (spin 1) fields, the interacting massive spin-2 field theory is necessarily a non-linear theory. The non-linearity originates from the physical requirement that a spin-2 field be coupled to the total energy-momentum tensor of the system. An illustrative example is the massless spin-2 field, better known as the gravitational field. The Einstein field equations are non-linear, and the non-linearity is due to the fact that the source of the gravitational field is the energy-momentum tensor of all fields including that of the gravitational field itself. We find that the problem of constructing the Lagrangian for an interacting massive spin-2 field essentially reduces to one of adding to the Einstein's field equation a (non-linear) mass term that is consistent with the constraint conditions that accompany any description of a higher-spin field. The past investigations[1-7] that have focussed on the problem of massive spin-2 field either (i) have ignored the problem of including the non-linear self-interaction terms of (ii) have not set up the mass term correctly.

Our second motivation relates to astrophysical considerations. To be specific, interest in pulsars and the behavior of matter at extremely high densities have led to an increasing attention to the many-body problem of relativistic neutron systems. When matter densities are higher than the nuclear matter density, short-range forces arising from the exchange of spin-2

f^o meson (1260 MeV) are expected to be non-negligible. The formalism that we develop in this paper is used in the following paper to derive a theory of relativistic neutrons interacting through a finite-range spin-2 field (in addition to scalar and vector fields). The resulting equation of state is found to possess several novel features that are due to the spin-2 interaction. The preliminary applications of these results in relation to the bulk properties of a neutron star have been reported by us in an earlier paper.[8]

The outline of this paper is as follows. In Section II we briefly review the properties of a free spin-2 field and recall that even for a free spin-2 field, there is a definite allowed form for the mass term that is consistent with the constraint conditions on the field. Section III describes the problems associated with a theory of interacting massive spin-2 field, and how we proceed to set up a consistent formalism for it. The derivation of the full Lagrangian is presented in Section IV. Section V discusses the problem of subsidiary condition in the non-linear theory and Section VI the structure of the source term in a classical situation. Subsequent papers will deal with the Hartree-approximation both for a high temperature gas of classical particles and for a zero temperature degenerate gas of fermions interacting through a spin-2 (tensor) field.

II. THE FREE SPIN-2 FIELD

A massive spin-2 field is described by a symmetric "tensor-potential" of rank two, $\phi_{\mu\nu}$. Such a tensor field has ten linearly independent components and in general is a mixture of fields corresponding to irreducible representations of spin-2, spin-1 and two spin-0 components. It will describe a spin-2 field only after we have removed the (3 + 1) - component vector contribution ($\partial^{\mu}\phi_{\mu\nu}$) and the scalar contribution (tr $\phi_{\mu\nu}$) by imposing the following constraint conditions:

$$\partial^{\mu}\phi_{\mu\nu} = 0 \tag{2.1}$$

$$\text{tr } \phi_{\mu\nu} \equiv \eta^{\mu\nu}\phi_{\mu\nu} = 0 \tag{2.2}$$

($\eta_{\mu\nu}$ = diag. (-1, 1, 1, 1).) When the constraint conditions (2.1) and (2.2) are satisfied, the residual five components will describe a pure spin-2 field.

As long as $\phi_{\mu\nu}$ represents a free field (i.e. not in interaction with other fields), a Lagrangian quadratic in the field variable and its derivatives can be constructed that gives the constraint conditions (2.1) and (2.2), and provides a linear equation of motion. Single-parameter Lagrange functions having such properties have been devised by Rivers[9] and Nath[3]. Bhargava and Watanabe[5] have formulated a 3-parameter Lagrange function from which they derived the field equation, the constraint conditions and also the condition that the field variable be symmetric in its indices. Prescription for constructing the generalized Lagrange function for a free system with arbitrary spin has also been given by Chang[10] who used the method of spin projection operators introduced by Fronsdal.[11]

The simplest form of the linear free field equations for the massive (mass μ) spin-2 field $\phi_{\mu\nu}$, from which the constraint conditions (2.1) and (2.2) follow, can be written as follows:

$$\mathcal{D}_{\mu\nu}^{\;\;\rho\sigma}\ \phi_{\rho\sigma} + \mu^2 F_{\mu\nu}^{\;\;\rho\sigma}\ \phi_{\rho\sigma} = 0$$

where

$$\begin{aligned}\mathcal{D}_{\mu\nu}^{\;\;\rho\sigma} &= -\ \partial^2\ \delta^\rho_\mu\ \delta^\sigma_\gamma + \partial_\mu\ \partial^\rho\ \delta^\sigma_\nu + \partial_\nu\ \partial^\sigma\ \delta^\rho_\mu \\ &\quad - \partial_\mu\ \partial_\nu\ \eta^{\rho\sigma} + \partial^2\ \eta_{\mu\nu}\ \eta^{\rho\sigma} - \eta_{\mu\nu}\ \partial^\rho\ \partial^\sigma \\ F^{\rho\sigma}_{\mu\nu} &= \delta^\rho_\mu\ \delta^\sigma_\nu - \eta_{\mu\nu}\ \eta^{\rho\sigma}\end{aligned} \tag{2.3}$$

and

$$\partial^2 \equiv \partial^\mu \partial_\mu$$

$$\phi = \mathrm{tr}\ \phi_{\mu\nu} \tag{2.4}$$

The quadratic (linearized) Lagrangian that yields Eq. (2.3) through $\frac{\delta\ L}{\delta\phi^{\mu\nu}} = 0$ is

$$\begin{aligned} L &= -\tfrac{1}{2}\ (\partial^\lambda\phi^{\mu\nu}\partial_\lambda\phi_{\mu\nu} + \mu^2\phi^{\mu\nu}\phi_{\mu\nu} - \partial^\lambda\phi\ \partial_\lambda\phi - \mu^2\phi^2) - \partial_\mu\phi^{\mu\nu}\partial_\nu\phi + \partial_\nu\phi^{\mu\nu}\partial^\lambda\phi_{\lambda\nu} \\ &= L^{(L)} + \mu^2\ L^{(PF)}\end{aligned} \tag{2.5}$$

Conditions (2.1) and (2.2) now follow from the dynamics of the theory. The divergence and trace of Eq. (2.3) provide, respectively

$$\mu^2(\partial^\rho\phi_{\rho\nu} - \partial_\nu\phi) = 0 \tag{2.6}$$

and

$$2(\partial^2\phi - \partial^\rho\partial^\sigma\phi_{\rho\sigma}) = 3\,\mu^2\phi \tag{2.7}$$

whose combinations are easily seen to yield (2.1) and (2.2). For this, however, the mass term in the Lagrangian must be of the form as appearing in Eq. (2.5) namely,

$$\mu^2 L^{(PF)} = -\tfrac{1}{2}\mu^2(\phi^{\mu\nu}\phi_{\mu\nu} - \phi^2) \tag{2.8}$$

This is usually referred to as the Pauli-Fierz Lagrangian for the spin-2 field.[12,13] It is easy to verify that with other choices for the mass term, e.g.

$$-\tfrac{1}{2}\mu^2(\phi^{\mu\nu}\phi_{\mu\nu} - a\phi^2) \quad , \qquad a \neq 1$$

one cannot retain the conditions (2.1) and (2.2) simultaneously. Thus, without the Pauli-Fierz form of the mass term, there will be admixtures of lower spin states which may give rise to negative-energy ghosts.

III. THE INTERACTING SPIN-2 FIELD

The linear massless spin-2 field must be coupled to a conserved symmetric tensor, which is identified with the energy-momentum tensor of the matter field, $t_{\mu\nu}$. This requirement is a consequence of the identity

$$\partial^{\mu}\mathcal{D}_{\mu\nu}{}^{\rho\sigma} = 0 \tag{3.1}$$

and the resulting equation is

$$\mathcal{D}_{\mu\nu}{}^{\rho\sigma}\ \phi_{\rho\sigma} = -\kappa\ t_{\mu\nu} \tag{3.2}$$

where κ characterizes the strength of the spin-2-field-matter-field coupling. The corresponding Lagrangian now can be written as

$$\begin{aligned} L &= L^{(L)} + \kappa L^{(I)} \\ L^{(I)} &= \tfrac{1}{2}\ \phi^{\mu\nu}\ t_{\mu\nu} \end{aligned} \tag{3.3}$$

It is well-known, however, that this coupling scheme, as it stands, is inconsistent: $t_{\mu\nu}$ is conserved only to the extent that the effect of the ϕ-field on the matter field is negligible; otherwise $\partial^{\mu}\ t_{\mu\nu} = 0$ is replaced by

$$\partial^{\mu}(t_{\mu\nu} + T_{\mu\nu}) = 0 \tag{3.4}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the ϕ-field. For this relationship to follow from the Lagrangian, the latter has to be modified. The result of this procedure, as shown by Gupta, Kraichnan, Wyss, Deser, and others,[14] is precisely the non-linear Einstein-equation,

$$\mathcal{D}_{\mu\nu}{}^{\rho\sigma}\ \phi_{\rho\sigma} = -\kappa\ A_{\mu}^{\rho}(\phi)\{t_{\rho\nu} + T_{\rho\nu}(\phi)\} \tag{3.5}$$

$$A_{\mu}^{\rho}(\phi) = \delta_{\mu}^{\rho} + 0\ (\phi^2)$$

The details of this and the origin of the $A_{\mu}{}^{\rho}(\phi)$ coefficient will be discussed in the next section. In a more conventional notation (3.5) is equivalent to

$$\begin{aligned} G_{\mu\nu}(\phi) &= -\frac{\kappa^2}{2}\,t_{\mu\nu} \\ G_{\mu\nu} &\equiv R_{\mu\nu} - \tfrac{1}{2}\,g_{\mu\nu}R \\ R_{\mu\nu} &= \Gamma_{\mu\beta}{}^{\alpha}\,\Gamma_{\alpha\nu}{}^{\beta} - \Gamma_{\mu\nu}{}^{\alpha}\,\Gamma_{\alpha\beta}{}^{\beta} - \partial_{\nu}\,\Gamma_{\mu\alpha}{}^{\alpha} - \partial_{\alpha}\,\Gamma_{\mu\nu}{}^{\alpha} \\ \Gamma_{\mu\nu}{}^{\alpha} &= \tfrac{1}{2}\cdot h^{\alpha\beta}(\partial_{\nu}g_{\beta\mu} + \partial_{\mu}g_{\beta\nu} - \partial_{\beta}g_{\mu\nu}) \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + \kappa\phi_{\mu\nu} \\ h^{\mu\rho}\ g_{\rho\nu} &= \delta_{\nu}^{\mu} \end{aligned} \tag{3.7}$$

Even though the equations have their well-known geometric interpretations, we wish to avoid emphasizing that aspect; $g_{\mu\nu}$ here is merely an abbreviation defined by (3.7); correspondingly we distinguish between $g_{\mu\nu}$ and its inverse $h^{\mu\nu}$ (<u>not</u> $g^{\mu\nu}$).

The Lagrangian that generates (3.6) through $\frac{1}{\sqrt{-|g|}}\,\frac{\delta\,\mathcal{L}}{\delta h^{\mu\nu}} = 0$ is the fámiliar Einstein Lagrangian $\mathcal{L}^{(E)}$ (the notation $\mathcal{L}$ is intended to underline the difference between the quadratic (linearized) Lagrangians $L(\phi)$ and the full non-linear Lagrangians $\mathcal{L}(g)$), combined with the interaction Lagrangian:

$$L^{(E)} = -\sqrt{-|g|}\,h^{\mu\nu}\,R_{\mu\nu} \tag{3.8}$$

$$L = \frac{1}{\kappa^2}\{L^{(E)} + \kappa^2 L^{(I)}\}$$

Now, turning to the problem of the coupling of the massive spin-2 field to the matter field, we observe that with a source $s_{\mu\nu}$

$$\mathcal{D}_{\mu\nu}^{\ \ \rho\sigma}\,\phi_{\rho\sigma} + \mu^2\,F_{\mu\nu}^{\ \ \rho\sigma}\,\phi_{\rho\sigma} = -\kappa\,s_{\mu\nu} \tag{3.9}$$

there is no need to require that $\partial^\mu s_{\mu\nu} = 0$; taking the divergence of (3.9) merely results in

$$\partial^\mu(\phi_{\mu\nu} - \eta_{\mu\nu}\,\phi) = -\frac{\kappa}{\mu^2}\,\partial^\mu\,s_{\mu\nu} \tag{3.10}$$

Taking the trace of (3.9) and substituting (3.6) in the resulting equation, generates the subsidiary conditions

$$\phi = \frac{\kappa}{3\mu^2}\{s + \frac{2}{\mu^2}\,\partial^\mu\partial^\nu\,s_{\mu\nu}\} \tag{3.11}$$

Thus, it appears that in contrast to the massless spin-2 theory, it is possible to construct a <u>linear</u>, massive spin-2 theory, without running into manifest inconsistencies (by choosing, for example,

$$s_{\mu\nu} = m\int d\tau\;\delta(x - \xi(\tau))\,\dot{\xi}_\mu(\tau)\,\dot{\xi}_\nu(\tau) \tag{3.12}$$

i.e. the energy-momentum tensor of the non-interacting field, as the source). There are, however, at least two difficulties with this approach. (i) even though there are no manifest inconsistencies in the theory, it is not evident at all that reasonable solutions of the field equations combined with the equation of motion for the particles can be obtained without violating the subsidiary conditions (3.11). (ii) The theory doesn't have any smooth limit for $\mu \to 0$: the massive source term $s_{\mu\nu}$ is completely different from the massless $A^{\rho}_{\mu}(t_{\rho\nu} + T_{\rho\nu})$ source.

The philosophy we adopt in the present paper is different. We will require that the theory, on the one hand, be a "smooth" continuation of the massless Einstein-theory, (in the restricted sense that in the $\mu = 0$ limit the Einstein field equations are recovered) and that, on the other hand, in the linear limit it reduce to the correct Pauli-Fierz formalism of massive spin-2 fields. Thus we add a mass term, say $\frac{1}{2} \mu^2 H_{\mu\nu}(\phi)$, to the Einstein equation. We note that since the fully interacting theory with zero mass is non-linear, there is no *a priori* reason to expect this mass term to be linear. Moreover, in the non-linear theory there exists no equivalent of the Pauli-Fierz mass criterion to serve as a guide for setting up the mass term. Since, however, the mass implies a finite range or an asymptotic fall-off of the forces, we shall take $H_{\mu\nu}$ to be independent of field derivatives. Thus, the equation of motion of the massive spin-2 field will now have the form:

$$G_{\mu\nu} + \tfrac{1}{2} \mu^2 H_{\mu\nu} = -\frac{\kappa^2}{2} t_{\mu\nu} \tag{3,13}$$

The corresponding Lagrangian that generates (3.13) through $\frac{1}{\sqrt{-|g|}} \frac{\delta L}{\delta g^{\mu\nu}} = 0$ now is

$$L = \frac{1}{\kappa^2} \{L^{(E)} + \mu^2 L^{(M)} + \kappa^2 L^{(I)}\} \tag{3.14}$$

where $L^{(M)}$ denotes the massive part of the Lagrangian. $L^{(M)}$ should be independent of field derivatives, and its variation with respect to the field will give $\frac{1}{2} H_{\mu\nu}$ (the $\frac{1}{2}$ factor is inserted for convenience).

At this point, it is important to note the essential difference between the massless and massive theories. The mass breaks the gauge invariance and, unlike the massless case, the massive theory doesn't lend itself to a (Riemannian) geometric interpretation. As far as the dynamics is concerned, however, we may draw analogies with the gravitational case, although such analogies must be considered as purely formal. All indices in the present formalism will be taken to indicate Lorentz indices. As we have pointed out already, we shall introduce different symbols for the "metric" ($g_{\mu\nu}$) and its inverse ($h^{\mu\nu}$). Raising or lowering of indices will be performed solely by using the Minkowski tensor $\eta_{\mu\nu}$, and not $h^{\mu\nu}$ or $g_{\mu\nu}$. We will also introduce

$$\bar{g}^{\mu\nu} \equiv \sqrt{-|g|}\, h^{\mu\nu}$$

$$|g| = \det g_{\mu\nu} = (\det h^{\mu\nu})^{-1} = \det \bar{g}^{\mu\nu} \tag{3.15}$$

$$g_{\mu\nu} = -\frac{1}{\sqrt{-|g|}} \text{ minor } \bar{g}^{\mu\nu}$$

$$\bar{g} = \text{tr } \bar{g}_{\mu\nu}$$

We already have

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa\, \phi_{\mu\nu} \tag{3.16a}$$

Analogously, we define $\bar{\phi}^{\mu\nu}$ by

$$\bar{g}^{\mu\nu} = \eta^{\mu\nu} + \kappa\, \bar{\phi}^{\mu\nu} \tag{3.16b}$$

Then, in the small field limit

$$\bar{\phi}^{\mu\nu} = \phi^{\mu\nu} - \tfrac{1}{2}\, \eta^{\mu\nu}\, \phi \tag{3.16c}$$

and

$$h^{\mu\nu} = \eta^{\mu\nu} - \kappa\, \phi^{\mu\nu} \tag{3.16d}$$

There is no a priori preference, at this point, as to which of these quantities is more appropriate to characterize the physical field.

To conclude this section, we mention earlier works aimed at constructing a general theory of an interacting massive spin-2 field. Ogievetsky and Polubarinov[4] have constructed Lagrangians, characterized by two parameters p and q, that lead to the subsidiary condition

$$\partial^{\mu}\phi_{\mu\nu} + q\, \partial_{\nu}\phi \quad = 0 \quad ,$$

instead of (2.1) and (2.2). Their source term, however, is not the total stress tensor $\theta_{\mu\nu}$ but rather

$$\theta_{\mu\nu} - \frac{1}{3\mu^2}\,(\partial_{\mu}\partial_{\nu} + \eta_{\mu\nu}\partial^2)\ \theta$$

The physical interpretation of such a source term is unclear. Besides, the massive part of the Lagrangian does not possess the required Pauli-Fierz criterion. Freund et al.,[15] whose approach is most similar to ours, have proposed a massive version of Einstein's gravitational equations; however, their theory also suffers from the non-compliance with the Pauli-Fierz limit for the mass term, and as such, it is also unsatisfactory in the linear limit. In a different context Salam and Strahdee[16] discussed the Lagrangian for the combination of Yang-Mills and tensor fields.

IV. DERIVATION OF THE LAGRANGIAN

To obtain the massive part of the Lagrangian for the linear Pauli-Fierz Lagrangian we follow the same path that leads to the Einstein Lagrangian from the linearized $L^{(L)}$. The principal milestones[14,15] along this path are now enumerated.

1) The "mixed" "tensor-density" is identified as the representation where the origin of non-linearity is manifest:

$$\sqrt{-|g|}\, h^{\alpha\gamma}\, G_{\gamma\beta} = \bar{G}^{(L)\alpha}{}_{\beta} + \frac{\kappa^2}{2}\, \bar{T}^{\alpha}_{\beta} \tag{4.1}$$

with

$$\bar{G}^{(L)\alpha\gamma} = \tfrac{1}{2}\, \bar{\mathcal{D}}^{\alpha\gamma}{}_{\rho\sigma}\, \bar{g}^{\rho\sigma}$$

$$\bar{\mathcal{D}}^{\alpha\gamma}{}_{\rho\sigma} = -\eta^{\alpha\gamma}\, \partial_\rho\, \partial_\sigma + \delta^\alpha_\sigma\, \partial^\gamma\, \partial_\rho + \delta^\gamma_\rho\, \partial^\alpha \partial_\sigma - \delta^\alpha_\sigma\, \delta^\gamma_\rho\, \partial^2 \tag{4.2}$$

obtained from (2.3) with the substitution (3.16b) and (3.16c).

2) $\bar{T}^{\alpha}_{\beta}$, the canonical stress tensor density, is

$$\begin{aligned} \bar{T}^\alpha_\beta &= \frac{1}{\kappa^2}\left\{ \partial_\beta h^{\mu\nu}\, \frac{\partial L^{(E)}}{\partial_\alpha h^{\mu\nu}} - \delta^\alpha_\beta\, \mathcal{L}^{(E)} + \bar{S}^\alpha_\beta \right\} \\ &= \frac{1}{\kappa^2}\left\{ \partial_\beta\, \bar{g}^{\mu\nu}\, \frac{\partial L^{(E)}}{\partial_\alpha \bar{g}^{\mu\nu}} - \delta^\alpha_\beta\, \mathcal{L}^{(E)} + \bar{S}^\alpha_\beta \right\} \\ &\equiv \bar{\mathcal{T}}^\alpha_\beta\, \mathcal{L}^{(E)} \end{aligned} \tag{4.3}$$

where $\bar{S}^\alpha_\beta$ is the symmetrizing Belifante term: it depends on the derivatives of the field only and its detailed structure is of no relevance for our purpose. $\bar{T}^{\alpha}_{\beta}$ is the operator defined by (4.3). The transformation from the $h^{\alpha\beta}$ -representation to $\bar{g}^{\alpha\beta}$ -representation is effected with the aid of the relation

$$d\bar{g}^{\alpha\beta} = \sqrt{-|g|}\ dh^{\alpha\beta} - \tfrac{1}{2} h^{\alpha\beta} g_{\mu\nu}\ dh^{\mu\nu} \tag{4.4}$$

3) $G_{\alpha\beta}$ and the source $t_{\alpha\beta}$ are derived from the variation of the Lagrangian $L = \frac{1}{\kappa^2}(L^{(E)} + \kappa^2 L^{(I)})$

$$\begin{aligned} G_{\alpha\beta} &= \frac{1}{\sqrt{-|g|}} \frac{\delta L^{(E)}}{\delta h^{\alpha\beta}} \\ \tfrac{1}{2} t_{\alpha\beta} &= \frac{1}{\sqrt{-|g|}} \frac{\delta L^{(I)}}{\delta h^{\alpha\beta}} \end{aligned} \tag{4.5}$$

4) As a result, the Einstein equation can be put in the form

$$\eta_{\beta\delta}\ \bar{D}^{\alpha\delta}_{\rho\sigma}\ \bar{g}^{\rho\sigma} = -\kappa^2 (\bar{T}^{\alpha}_{\beta} + \sqrt{-|g|}\ h^{\alpha\gamma}\ t_{\gamma\beta}) \tag{4.6}$$

with the Lagrangian satisfying the differential equation

$$h^{\alpha\delta} \frac{\delta L^{(E)}}{\delta h^{\delta\beta}} = \tfrac{1}{2} \eta_{\gamma\beta}\ \bar{D}^{\alpha\gamma}_{\rho\sigma}\ \bar{g}^{\rho\sigma} + \frac{\kappa^2}{2}\ \bar{T}^{\alpha}_{\beta}\ L^{(E)} \tag{4.7}$$

The massive term is now to be characterized by the additional term in the Lagrangian $L^{(M)}$ $\left[L = \frac{1}{\kappa^2}\left(L^E + \mu^2 L^{(M)} + \kappa^2 L^{(I)}\right)\right]$, and the additional term in the equation of motion, $\frac{1}{2} H_{\alpha\beta}$. The equivalents of (1) through (4) above now lead to the following steps.

1) The massive contribution $H_{\alpha\beta}$ is split into the linear part plus the massive stress tensor density:

$$\mu^2 \sqrt{-|g|}\ h^{\alpha\gamma}\ H_{\gamma\beta} = \mu^2\ \bar{H}^{(L)\alpha}_{\ \ \ \ \beta} + \kappa^2\ \bar{T}^{(M)\alpha}_{\ \ \ \ \ \beta} \tag{4.8}$$

with

$$\bar{H}^{(L)\alpha\gamma} = \bar{F}^{\alpha\gamma}_{\rho\sigma} (\bar{g}^{\rho\sigma} - \eta^{\rho\sigma})$$
$$\bar{F}^{\alpha\gamma}_{\rho\sigma} = \delta^\alpha_\rho \, \delta^\gamma_\sigma + \tfrac{1}{2} \eta^{\alpha\gamma} \eta_{\rho\sigma} \tag{4.9}$$

obtained from (2.3) with the substitution (3.16b) and (3.16c).

2) $\mu^2 \bar{T}^{(M)\alpha}{}_\beta$, the canonical massive stress tensor density is given by

$$\bar{T}^{(M)\alpha}{}_\beta = \bar{T}^\alpha_\beta \, L^{(M)} = - \frac{\mu^2}{\kappa^2} \, \delta^\alpha_\beta \, L^{(M)} \tag{4.10}$$

3) $H_{\alpha\beta}$ is derived from the variation of the massive Lagrangian,

$$\tfrac{1}{2} H_{\alpha\beta} = \frac{1}{\sqrt{-|g|}} \frac{\delta L^{(M)}}{\delta h^{\alpha\beta}} = \frac{1}{\sqrt{-|g|}} \frac{\partial L^{(M)}}{\partial h^{\alpha\beta}} \tag{4.11}$$

4) The full massive field equation can now be put in the form

$$\eta_{\beta\delta} \left\{ \bar{D}^{\alpha\delta}_{\rho\sigma} + \mu^2 \, \bar{F}^{\alpha\delta}_{\rho\sigma} \, (\bar{g}^{\rho\sigma} - \eta^{\rho\sigma}) \right\} \tag{4.12}$$
$$= - \kappa^2 \, (\bar{T}^\alpha_\beta + \sqrt{-|g|} \;\; h^{\alpha\gamma} \, t_{\gamma\beta})$$
$$\bar{T}^\alpha_\beta = \bar{T}^{(E)\alpha}{}_\beta + \mu^2 \, \bar{T}^{(M)\alpha}{}_\beta$$

with the massive Lagrangian satisfying the differential equation

$$h^{\alpha\gamma} \frac{\partial L^{(M)}}{\partial h^{\gamma\beta}} = \tfrac{1}{2} \eta_{\beta\gamma} \, \bar{F}^{\alpha\gamma}_{\rho\sigma} \, (\bar{g}^{\rho\sigma} - \eta^{\rho\sigma}) - \tfrac{1}{2} \delta^\alpha_\beta \, L^{(M)} \tag{4.13}$$

or, using (4.4) and (4.9)

$$2\bar{g}^{\alpha\gamma} \frac{\partial L^{(M)}}{\partial \bar{g}^{\gamma\beta}} - \delta^\alpha_\beta \, \frac{\partial L^{(M)}}{\partial \bar{g}^{\rho\sigma}} \, \bar{g}^{\rho\sigma} = \bar{g}^\alpha_\beta - 3\delta^\alpha_\beta + \tfrac{1}{2} \delta^\alpha_\beta \, \bar{g} - \delta^\alpha_\beta L^{(M)} \tag{4.14}$$

For reasons to be explained below, we generalize (4.12), by allowing for the massive part of the stress tensor to carry an additional weight, say z, leading to the replacment of the total field stress tensor density as

$$\bar{T}^{\alpha}_{\beta} \rightarrow \bar{T}^{(E)\alpha}_{\beta} + (1 + z)\mu^2\ \bar{T}^{(M)\alpha}_{\beta} = \bar{T}^{\alpha}_{\beta} + z\mu^2\ \bar{T}^{(M)\alpha}_{\beta} \tag{4.15}$$

and resulting in

$$2\bar{g}^{\alpha\gamma}\ \frac{\partial L^{(M)}}{\partial \bar{g}^{\gamma\beta}} - \delta^{\alpha}_{\beta}\ \frac{\partial L^{(M)}}{\partial \bar{g}^{\rho\sigma}}\ \bar{g}^{\rho\sigma} = \bar{g}^{\alpha}_{\beta} - 3\delta^{\alpha}_{\beta} + \tfrac{1}{2}\ \delta^{\alpha}_{\beta}\ \bar{g} - \delta^{\alpha}_{\beta}(1 + z)\ L^{(M)} \tag{4.16}$$

as the basic equation for the determination of $L^{(M)}$.

In principle, $L^{(M)}$, which does not depend on the derivatives of the field, can depend on any invariant that we can construct out of the field components. As noted by Freud et al.,[15] one such invariant is $p \equiv \sqrt{-|g|}$; should $L^{(M)}$ depend only p, then Eq. (4.16) could not be obeyed (because of the presence of the term $\bar{g}^{\alpha}_{\beta}$); in addition to p, $L^{(M)}$ has to depend on at least one more Lorentz-invariant quantity: the simplest such quantity is $\bar{g} \equiv \mathrm{tr}\bar{g}^{\mu\nu} = \eta_{\mu\nu}\ \bar{g}^{\mu\nu}$. We shall thus assume that $L^{(M)}$ depends on $\bar{g}^{\mu\nu}$ only through the invariant combinations p and $\bar{g}$.

Noting that

$$\frac{\partial p}{\partial \bar{g}^{\mu\nu}} = \tfrac{1}{2}\ g_{\mu\nu}$$

$$\frac{\partial \bar{g}}{\partial \bar{g}^{\mu\nu}} = \eta_{\mu\nu} \tag{4.17}$$

and consequently

$$2\bar{g}^{\alpha\gamma}\ \frac{\partial L^{(M)}}{\partial \bar{g}^{\gamma\beta}} = 2\bar{g}^{\alpha\gamma}\left(\tfrac{1}{2}\ g_{\gamma\beta}\ \frac{\partial L^{(M)}}{\partial p} + \eta_{\gamma\beta}\ \frac{\partial L^{(M)}}{\partial \bar{g}}\right)$$

$$= \delta^{\alpha}_{\beta}\ p\ \frac{\partial L^{(M)}}{\partial p} + 2\bar{g}^{\alpha}_{\beta}\ \frac{\partial L^{(M)}}{\partial \bar{g}} \tag{4.18}$$

we have

$$\frac{\partial L^{(M)}}{\partial \bar{g}^{\rho\sigma}} \bar{g}^{\rho\sigma} = \left(\tfrac{1}{2} g_{\rho\sigma} \frac{\partial L^{(M)}}{\partial p} + \eta_{\rho\sigma} \frac{\partial L^{(M)}}{\partial q} \right) \bar{g}^{\rho\sigma}$$

$$= 2p \frac{\partial L^{(M)}}{\partial p} + \bar{g} \frac{\partial L^{(M)}}{\partial \bar{g}} \quad , \tag{4.19}$$

By using (4.16) we further obtain

$$\delta^\alpha_\beta \left(\tfrac{1}{2} \bar{g} - 3 - (1 + z) L^{(M)} + \bar{g} \frac{\partial L^{(M)}}{\partial \bar{g}} + p \frac{\partial L^{(M)}}{\partial p} \right) + \bar{g}^\alpha_\beta \left(1 - 2 \frac{\partial L^{(M)}}{\partial \bar{g}} \right) = 0 \tag{4.20}$$

For Eq. (4.20) to have a solution the coefficients of both δ^α_β and $\bar{g}^\alpha_\beta$ have to vanish. The latter requirement yields

$$L^{(M)} = \tfrac{1}{2} \bar{g} + K(p) \tag{4.21}$$

where K(p) is still to be determined. (4.21) combined with (4.20) leads to

$$-3 + \bar{g} \{1 - \tfrac{1}{2}(1 + z)\} - (1 + z) K(p) + p \frac{dK(p)}{dp} = 0 \tag{4.22}$$

(4.22) is consistent only if the coefficient of $\bar{g}$ is 0, yielding

$$z = 1 \tag{4.23}$$

At this point the justification of introducing the renormalized weight (1 + z) emerges. We see that the naive choice z = 0 leads to inconsistencies. On the other hand, the choice z = 1, signifying in effect that the massive part of the field energy-momentum couples with a double strength to the field itself, generates a perfectly reasonable solution: in particular, the linear limit has the physically required Pauli-Fierz form. Apart from the consistency requirement we cannot, at this point, provide an intuitively simple explanation of the doubling effect - although we suspect the existence of a simple physical picture.

Returning now to (4.22), its solution with $z = 1$ is immediate:

$$K(p) = -\frac{3}{2} + kp^2$$
$$L^{(M)} = -\frac{3}{2} + kp^2 + \tfrac{1}{2}\bar{g} \qquad (4.24)$$

The constant of integration, k, can be fixed by requiring that at $\bar{g}^{\alpha\beta} = \eta^{\alpha\beta}$ at $L^{(M)} = 0$. This way we find $k = -\tfrac{1}{2}$ and

$$L^{(M)} = \tfrac{1}{2}\,(\bar{g} + |g| - 3) \qquad (4.25)$$

It is perhaps more illuminating (and more pleasing) to rewrite (4.25) as

$$L^{(M)} = \tfrac{1}{2}\,\{1 + \kappa\,\bar{\phi} + \det\,(\eta^{\mu\nu} + \kappa\,\bar{\phi}^{\mu\nu})\} \qquad (4.26)$$

The resulting massive term in the field equation (3.13) is calculated from (4.25) with the aid of (4.11) or (4.8), (4.9) and (4.10):

$$H_{\mu\nu} = \frac{1}{\sqrt{-|g|}}\, g_{\mu}{}^{\rho}\,\{\bar{g}_{\rho\nu} - \eta_{\rho\nu}(|g| + \tfrac{1}{2}\bar{g})\} \qquad (4.27)$$

V. THE SUBSIDIARY CONDITION

In this section we describe and discuss the subsidiary conditions of the nonlinear theory. It is convenient to use (4.12) combined with (4.15) as a starting point. Taking the divergence of the combined equation, one finds,

$$\mu^2\{\partial_\mu \bar{g}^{\mu\nu} + \tfrac{1}{2}\,\partial^\nu \bar{g}\} = -\,z\kappa^2\mu^2\partial_\mu\, T^{(M)\mu}{}_{\nu}$$

$$= +\,\frac{z}{2}\,\mu^2\partial^\nu(\bar{g} + |g|) \tag{5.1}$$

that is

$$\partial_\mu\, \bar{g}^{\mu\nu} - \frac{z}{2}\,\partial_\mu |g| + \tfrac{1}{2}\,(1 - z)\,\partial^\nu \bar{g} = 0 \tag{5.2}$$

We note that in the linear theory

$$\begin{aligned} |g| &= -\,(1 + \kappa\bar{\phi}) \\ \bar{g} &= 4 + \kappa\bar{\phi} \end{aligned} \tag{5.3}$$

and the divergence condition reduces to the familiar (2.6), irrespective of the value of z. z = 0 would leave this linear condition unaltered. With the required z = 1 conditions observed, however, (5.1) becomes

$$\partial_\mu \bar{g}^{\mu\nu} - \tfrac{1}{2}\,\partial^\nu |g| = 0 \tag{5.4}$$

(5.4), even though it consitutes a set of non-linear differential equations, is not substantially different from (5.2) and can serve to eliminate 4 degrees of freedom.

Turning now to the trace condition

$$-2\,\{\partial^\rho\partial^\sigma\, \bar{g}_{\rho\sigma} + \tfrac{1}{2}\,\partial^2\, \bar{g}\} + 3\mu^2(\bar{g} - 4) = -\,\kappa^2\{\bar{T} + \bar{t}^* + z\bar{T}^{(M)}\} \tag{5.5}$$

$$\bar{T} = \bar{T}^\alpha_\alpha \quad \text{etc.}$$

$$\bar{t}^* = h^{\alpha\beta}\, \bar{t}_{\alpha\beta}$$

one observes that by using (5.4), (5.5) can be rewritten as

$$z\partial^2(\bar{g} + |g|) + \mu^2\{2z|g| + (2z - 3)\bar{g} + 6(z - 2)\} = - \kappa^2(\bar{T} + \bar{t}^*) \tag{5.6}$$

Again, in the linear limit one recovers the condition (2.7) which remains unaltered for $z = 0$. However, for $z = 1$ one obtains

$$\partial^2(\bar{g} + |g|) + \mu^2(2|g| - \bar{g} + 6) = - \kappa^2(\bar{T} + \bar{t}^*) \tag{5.7}$$

Thus, because of the appearance of the second order time derivative, (5.7) ceases to be a constraint condition on the full non-linear theory, and the 6th degree of freedom remains inextricably coupled to the 5 spin-2 degrees.

VI. THE SOURCE TERM

In order to exhibit the details of the way the spin-2 field couples to the matter-field, we discuss in this section the structure of the classical stress tensor that appears as the source in Eq. (4.12). The aspects of the quantum situation where a matter field is coupled to the spin-2 field will be discussed in a separate paper.

The source term in (4.12), say $-\kappa^2\,\bar{t}^{*\mu}_{\ \nu}$, can be expressed with the aid of (4.5):

$$\begin{aligned} -\kappa^2\bar{t}^{*\mu}_{\ \nu} &= -\kappa^2\sqrt{-|g|}\,h^{\alpha\gamma}\,t_{\gamma\beta} \\ &= -2\kappa^2\,h^{\alpha\gamma}\,\frac{\partial L^{(I)}}{\partial h^{\gamma\beta}} \\ &= -2\kappa^2\,\frac{\partial L^{(I)}}{\partial g_{\alpha\gamma}}\,g_{\gamma\beta} \end{aligned} \tag{6.1}$$

In the choice of $L^{(I)}$ we are guided again by the general relativistic analogy, and set

$$L^{(I)} = -m\int\sqrt{-u^{\rho}\,g_{\rho\sigma}\,u^{\sigma}}\;\delta(x-\xi(\tau))d\tau \tag{6.2}$$

Here m is the mass of the source particle, $\xi(\tau)$ its trajectory and $u^{\sigma} = d\xi^{\sigma}/d\tau$. The source term thus becomes

$$\bar{t}^{*\alpha}_{\ \beta} = +M\nu\,u^{\alpha}u^{\gamma}g_{\gamma\beta} \tag{6.3}$$

$u^{\alpha} \equiv u^{\alpha}(x)$ is now to be understood as a hydrodynamic variable, $\nu = \nu(x)$ is the invariant density

$$\nu(x) = \int\delta(x-\xi(\tau))\,d\tau \tag{6.4}$$

and M

$$M = \frac{m}{\sqrt{-u^\alpha g_{\alpha\beta} u^\beta}} \tag{6.5}$$

can be interpreted as the variable rest mass. This interpretation is suggested by the generalized Lagrangian formalism for relativistic particles.[17] Writing $L^{(I)}$ in the form

$$\begin{aligned} L^{(I)} &= \int \Lambda(u)\ \delta(x - \xi(\tau))\ d\tau \\ &= -m \sqrt{1 - \kappa\ u^\alpha \phi_{\alpha\beta} u^\beta} \end{aligned} \tag{6.6}$$

the variable rest mass is given by[17]

$$M = \frac{\partial\Lambda}{\partial u^\alpha}\ u^\alpha - \Lambda \tag{6.7}$$

leading indeed to (6.5).

The (canonical) particle momentum is, in the same formalism,

$$\begin{aligned} p_\mu &= Mu_\mu + \frac{\partial\Lambda}{\partial u^\mu} \\ &= Mg_{\mu\nu} u^\nu \end{aligned} \tag{6.8}$$

which allows one to write (6.3) in the form

$$\bar{t}{}^{*\alpha}_{\ \beta} = \nu\ u^\alpha\ p_\beta \tag{6.9}$$

p_μ so defined now has the property that it generates the equation of motion of the particle through

$$\begin{aligned} \frac{dp_\mu}{d\tau} &= \frac{\partial\Lambda}{\partial x^\mu} \\ &= \frac{\kappa}{2} M\ u^\alpha \frac{\partial\phi_{\alpha\beta}}{\partial x^\mu}\ u^\beta \end{aligned} \tag{6.10}$$

This is, of course, only an alternative expression of the general energy-momentum conservation $\partial_\mu(\bar{T}^\mu_\nu + \bar{t}^{*\mu}_\nu) = 0$.

It is instructive to consider the linearized versions of the relations derived in this section. (6.2) becomes

$$L^{(I)} = - m\int \left\{1 - \frac{\kappa}{2} u^\alpha \phi_{\alpha\beta} u^\beta \right\} \delta(x - \xi(\tau))\, d\tau \tag{6.11}$$

which represents the most natural coupling scheme for a linear tensor field. The linearized equivalents of (6.5) and (6.7) are[17]

$$M = m(1 + \frac{\kappa}{2} u^\alpha \phi_{\alpha\beta} u^\beta) \tag{6.12}$$

$$p_\mu = M u_\mu + \kappa\, m\, \phi_{\mu\nu}\, u^\nu$$

We observe that by introducing the new field variable $\Phi^{\mu\nu}$ through the re-scaling

$$\Phi^{\mu\nu} = \tfrac{1}{2}\, \kappa\, \frac{m}{f}\, \phi^{\mu\nu} \tag{6.13}$$

where f is the "charge" associated with the tensor field, the equation of motion takes a more familiar form; $\Phi^{\mu\nu}$ now plays the role of the tensor-potential. In the field equation the identification

$$\kappa^2 = 16\, \pi\, \frac{f^2}{m^2} \tag{6.14}$$

is necessary to arrive at the generalization of the customary structure.

The important conclusion emerging from this section is that in addition to the manifest non-linearities of the field equation, further non-linearities appear through the implicit $g_{\mu\nu}$ dependence of the source term, which in turn, can be traced to the field dependence of the particle rest mass and momenta.

A further conclusion, which will be exploited in a subsequent paper, is that in the case the interaction is mediated by scalar, vector, etc., particles in addition to the spin-2 mesons, the coupling through the energy momentum tensor requires a "universal" coupling of the spin-2 field to the other (scalar, vector, etc.) fields with the same coupling strength as free matter. This, and the implication of such a theory on the three fundamental tests of relativity, are being studied.

VII. CONCLUSIONS

In this paper we have constructed a Lagrangian theory of particles interacting through a massive spin-2 (tensor) field. The Lagrangian and the field equation are found to be non-linear, the non-linearity being principally the manifestation of the dynamical requirement that the spin-2 field couple to the total energy-momentum tensor of the system. This at the same time, implies that the spin-2 field should couple to all other fields with a universal coupling strength. We point out that the dynamics of the full non-linear theory dictates that an extra "ghostlike" scalar degree of freedom always accompany the spin-2 field. However, this does not necessarily lead to the energy being unbounded from below (as to be demonstrated in the subsequent papers). The main merit of the theory presented here is that in the limit of weak coupling, where one is in a position to make definitive statements about the excitations and about the boundedness of the energy, all spurious excitations we avoided, and the formalism is fully consistent with the constraints on the field variables. This is a minimum requirement for any consistent interacting massive spin-2 theory, and in this respect our results represent the first satisfactory formulation of such a theory. This is a departure from all previous formulations; these latter do not comply with the Pauli-Fierz criterion, and so remain unsatisfactory even at the limit of weak coupling.

Our field equations bear obvious similarity to the Einstein gravitational equations. However, the mass term has no relationship to the cosmological term of the Einstein theory. If viewed as a theory of gravitation the theory thus violates the equivalence principle, and hence no "general covariance" is implied by it; neither is it devoid of all the difficulties plaguing "finite range" gravitational theories.[13] Nevertheless, in view of the problems that are associated with quantizing Einstein's theory, one may question the requirement of general covariance. In that case, the present formalism can be viewed from the point of setting up an acceptable theory of a finite-range gravitation. The cosmological ramifications of such an attempt, however, remain to be seen.

REFERENCES

1. S. K. Kundu, Nuovo Cimento 31, 918 (1964).
2. T. Ino, M. Matsuda and S. Sawada, Prog. Theor. Phys. 33, 489 (1965).
3. L. M. Nath, Nucl. Phys. 68, 660 (1965).
4. V. I. Ogievetsky and I. V. Polubarinov, Ann. Phys. (NY) 35, 167 (1965).
5. S. C. Bhargava and H. Watanabe, Nucl. Phys. 87, 273 (1966).
6. H. van Dam and M. Veltman, Nucl. Phys. B22 297 (1970).
7. A. K. Nagpal, Lett. Nuovo Cimento 8, 353 (1973), and references therein.
8. V. Canuto, B. Datta and G. Kalman, Ap. J. 221, 274 (1978).
9. R. J. Rivers, Nuovo Cimento 34, 386 (1964).
10. S.-J. Chang, Phys. Rev. 161, 1308 (1967).
11. C. Fronsdal, Nuovo Cimento Suppl. 9, 416 (1958).
12. M. Fierz and W. Pauli, Proc. Roy. Soc. A173, 211 (1939).
13. D. G. Boulware and S. Deser, Phys. Rev. D 6, 3368 (1972).
14. S. Deser, Jour. Gen. Rel. Grav. 1, 9 (1970); S. N. Gupta, Phys. Rev. 96, 1683 (1954); Proc. Phy. Soc (London) A65, 608 (1952); Rev. Mod. Phys. 29, 334 (1957); R. H. Kraichnan, Phys. Rev. 98, 1118 (1955); W. Wyss, Helv. Phys. Acta 38, 469 (1965).
15. P. G. O. Freund, A. Maheshwari and E. Schonberg, Ap. J. 157, 857 (1969).
16. A. Salam and J. Strathdee, Phys. Rev. D 14, 2830 (1976).
17. G. Kalman, Phys. Rev. 123, 384 (1961); W. E. Thirring, Annals Phys. (NY) 16, 96 (1961); P. Mittelstadt and J. B. Barbour, Zeitschrift f. Physik 203, 82 (1967).